HW 12

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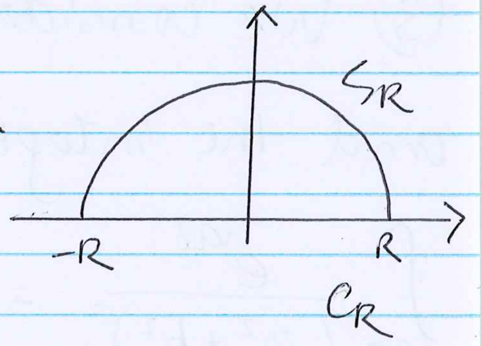
(2) $$\int_{C_R} \frac{dz}{(z^2+1)^2} = \int_{S_R} \frac{dz}{(z^2+1)^2} + \int_{-R}^{R} \frac{dx}{(x^2+1)^2}$$

$$\left| \int_{S_R} \frac{dz}{(z^2+1)^2} \right| \le \frac{\pi R}{(R^2-1)^2} \quad \text{for large } R.$$



$$\to 0 \quad \text{as } R \to \infty$$

$\cancel{\int_{C_R} d}$ $\quad (z^2+1)^2 = 0$ iff $z = \pm i$

$$\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2} = \lim_{z \to i} \frac{d}{dz} \frac{(z-i)^2}{(z^2+1)^2}$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^3}$$

$$= \frac{-i}{4}$$

$$\frac{\pi}{2} = \int_{C_R} \frac{dz}{(z^2+1)^2} = \int_{S_R} \frac{dz}{(z^2+1)^2} + \int_{-R}^{R} \frac{dx}{(x^2+1)^2}$$

$$\frac{\pi}{4} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} \quad \text{as } R \to \infty.$$

(4) Refer to Tutorial 10 question (1)

(9) Refer to tutorial 9 question (1)

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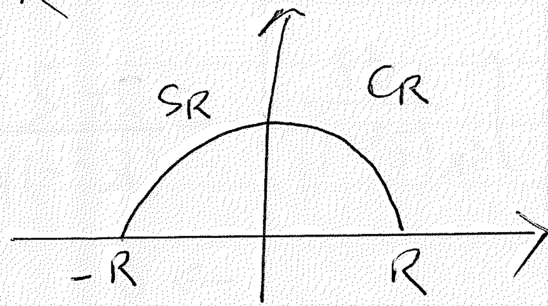
(3) We consider the contour C_R



and the integral

$$\int_{C_R} \frac{e^{iaz}}{(z^2+b^2)^2} = \int_{S_R} \frac{e^{iaz}}{(z^2+b^2)^2} + \int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}$$

$$\left| \int_{S_R} \frac{e^{iaz}}{(z^2+b^2)^2} \right| \le \frac{\pi}{a(R^2-b^2)^2}$$ for large enough b.

$\to 0$ as $R \to \infty$ by Jordan Lemma.

$$\operatorname{Res}_{z=bi} \frac{e^{iaz}}{(z^2+b^2)^2} = \lim_{z\to bi} \frac{d}{dz} \frac{e^{iaz}(z-bi)^2}{(z^2+b^2)^2}$$

$$= \lim_{z\to bi} \frac{(z+bi)^2 ia e^{iaz} - e^{iaz} 2(z+bi)}{(z+bi)^4}$$

$$= -\frac{ie^{-ab}(ab+1)}{4b^3}$$

Thus $\frac{+\pi e^{-ab}(ab+1)}{2b^3} = \int_{C_R} \frac{e^{iaz}}{(z^2+b^2)^2}$

$$= \int_{S_R} \frac{e^{iaz}}{(z^2+b^2)^2} + \int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}$$

Taking $R \to \infty$, we have

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} = \frac{\pi(1+ab)e^{-ab}}{4b^3}$$

(5) Consider the same contour in above,

$$\int_{C_R} \frac{z^3 e^{iaz}}{z^4+4} = \int_{S_R} \frac{z^3 e^{iaz}}{z^4+4} + \int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}$$

$$\left| \int_{S_R} \frac{z^3 e^{iaz}}{z^4+4} \right| \leq \frac{\pi R^3}{(R^4-4)a} \to 0 \text{ as } R \to \infty$$

by Jordon Lemma.

$z^4+4=0 \Rightarrow z = \pm 1 \pm i$, in which only $1+i$ and $-1+i$ are inside C_R. Since they are simple pole, the residue can be calculated by

$$\operatorname*{Res}_{z=1+i} \frac{z^3 e^{iaz}}{z^4+4} = \lim_{z=1+i} \frac{z^3 e^{iaz}(z-(1+i))}{z^4+4} = \frac{e^{(-1+i)a}}{4}$$

Similarly $\displaystyle \operatorname*{Res}_{z=-1+i} \frac{z^3 e^{iaz}}{z^4+4} = \frac{e^{(-1-i)a}}{4}$

Therefore $\displaystyle \frac{\pi i}{2}\left(e^{(-i-1)a} + e^{(-1+i)a} \right) = \int_{-\infty}^{\infty} \frac{z^3 e^{iaz}}{z^4+4}$

as $R \to \infty$.

Comparing the imaginary part,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a.$$

(8) Similarly in (5), Ans $= -\dfrac{\pi}{e}\sin 2$.

(12) Similarly in (3).

$$\left| \int_{S_R} \frac{e^{iz}}{(z+a)^2+b^2} \right| \to 0 \quad \text{by Jordan Lemma.}$$

as $R \to \infty$.

$$(z+a)^2 + b^2 = 0$$

$$(z+a)^2 = b^2 e^{i\pi}$$

$$z = b e^{i\pi/2} - a \text{ or}$$

$$b e^{3i\pi/2} - a$$

$$= bi - a \text{ or } -ib - a.$$

Since $z = bi - a$ is simple pole,

$\pi e^{-1}\cos 1$

$$\operatorname*{Res}_{z=-a+bi} \frac{e^{iz}}{(z+a)^2+b^2} = \lim_{z \to -a+bi} \frac{e^{iz}(z-(-a+bi))}{(z+a)^2+b^2}$$

$$= \lim_{z \to -a+bi} \frac{e^{iz} + ie^{iz}(z-(-a+bi))}{2(z+a)}$$

$$= \frac{e^{-b-ai}}{2bi}$$

Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x+a)^2+b^2} = \frac{\pi}{b} e^{-b} \cos a$

12a It's because exp is analytic,

$\cancel{\oint}$

12b $\displaystyle\int_{C_R} e^{iz^2} dz = \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} Rie^{i\theta} d\theta$

$$\left|\int_{C_R} e^{iz^2} dz\right| \le \frac{R}{2}\int_0^{\pi/2} \left|e^{iR^2 e^{i\phi}}\right| d\phi$$

$$= \frac{R}{2}\int_0^{\pi/2} e^{-R^2 \sin\phi} d\phi.$$

~~12c~~ $$\le \frac{R}{2} \frac{\pi}{2R^2}$$

12c Immediately by a and b.

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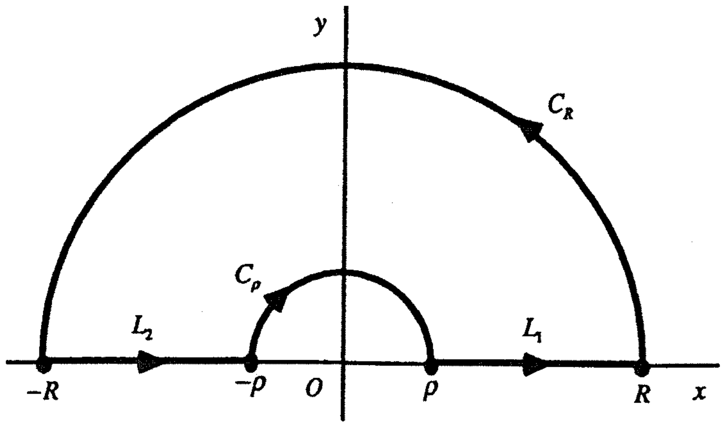
1. The main problem here is to derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z)=\frac{e^{iaz}-e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_\rho} f(z)\,dz=0,$$

or

$$\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz=-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1: z=re^{i0}=r\ (\rho\le r\le R)\quad\text{and}\quad -L_2: z=re^{i\pi}=-r\ (\rho\le r\le R),$$

we can see that

$$\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz=\int_{L_1} f(z)\,dz-\int_{-L_2} f(z)\,dz=\int_\rho^R\frac{e^{iar}-e^{ibr}}{r^2}dr+\int_\rho^R\frac{e^{-iar}-e^{-ibr}}{r^2}dr$$

$$=\int_\rho^R\frac{(e^{iar}+e^{-iar})-(e^{ibr}+e^{-ibr})}{r^2}dr=2\int_\rho^R\frac{\cos(ar)-\cos(br)}{r^2}dr.$$

Thus

$$2\int_\rho^R\frac{\cos(ar)-\cos(br)}{r^2}dr=-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho\to 0$, we write

$$f(z)=\frac{1}{z^2}\left[\left(1+\frac{iaz}{1!}+\frac{(iaz)^2}{2!}+\frac{(iaz)^3}{3!}+\cdots\right)-\left(1+\frac{ibz}{1!}+\frac{(ibz)^2}{2!}+\frac{(ibz)^3}{3!}+\cdots\right)\right]$$

$$=\frac{i(a-b)}{z}+\cdots\quad(0<|z|<\infty).$$

From this we see that $z=0$ is a simple pole of $f(z)$, with residue $B_0=i(a-b)$. Thus

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz=-B_0\pi i=-i(a-b)\pi i=\pi(a-b).$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if z is a point on C_R, then

$$f(z) \leq \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \leq \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_0^\infty \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a=0$ and $b=2$, that result becomes

$$\int_0^\infty \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1-2\sin^2 x$, and we arrive at

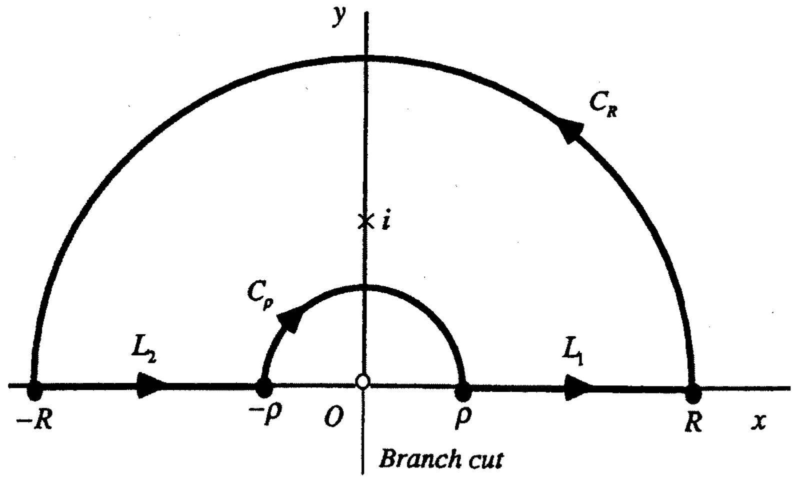
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

2 Let us first use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since

$$L_1 : z = re^{i0} = r \; (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \; (\rho \le r \le R),$$

we may write

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} - i\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = (1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Now the point $z=i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{1}{2i}\left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi\sqrt{R}}{(R^2-1)} = \frac{\pi}{\sqrt{R}\left(R-\frac{1}{R}\right)} \to 0 \text{ as } R \to \infty.$$

Finally, then, we have

$$(1-i)\int_0^\infty \frac{dr}{\sqrt{r}(r^2+1)} = \frac{\pi(1-i)}{\sqrt{2}},$$

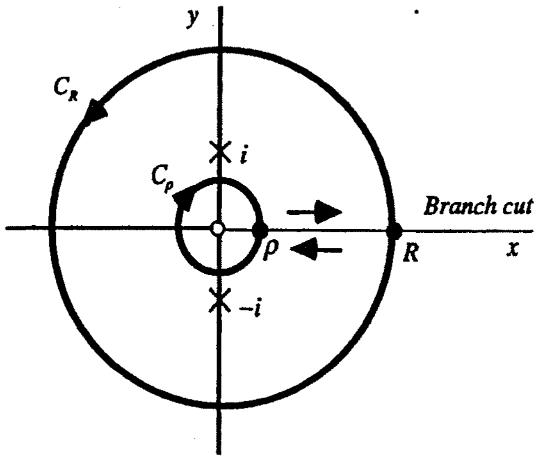
which is the same as

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

3 To evaluate the improper integral $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad (|z|>0, 0<\arg z<2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 99 in Sec. 77. We stipulate that $\rho<1$ and $R>1$, so that the singularities $z=\pm i$ are between C_ρ and C_R.



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~~Refer to (1) and (2) is are refered to the next document~~

(5) $\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$

$$= \int_\infty^0 \left(\frac{1}{x+1}\right)^{p-1} \left(\frac{x}{x+1}\right)^{q-1} \frac{-1}{(x+1)^2}\,dx$$

$$= \int_0^\infty \frac{x^{q-1}}{(x+1)^{p+q}}\,dx$$

If $q = 1 - p$

$$\int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \int_0^\infty \frac{x^{-p}}{x+1}\,dx$$

$$= \frac{\pi}{\sin(p\pi)}.$$